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ON THE STABILITY OF A PLANE COUETTE FLOW

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Iu. B. PONOMARENKO
(Moscow)

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Sufficient stability conditions (1.8), (1.10) are defined. Stability for large Reynolds numbers R is analyzed by asymptotic and numerical methods; it is shown that the flow is stable for $R \to \infty$

1. The stability of plane Couette flow is determined by the eigenvalues of the problem condidered in [1]

$$(D^2-\alpha^2)^2\varphi - i\alpha R(y-c)(D^2-\alpha^2)\varphi = 0 \qquad \left(D = \frac{d}{dy}\right) \qquad (1.1)$$
$$D\varphi(\pm 1) = \varphi(\pm 1) = 0 \qquad (-1 \leqslant y \leqslant 1)$$

The flow is stable if for any values of the Reynolds number R and of the wave number α, all of the eigenvalues $c = c_r + ic_i$ have a negative imaginary part.

Investigators [2 – 8] of the problem (1.1) assumed the flow to be stable; this assumption had not been completely substantiated thus far, however, because either particular values of parameters R and α, or special eigenvalues only had been considered. The particular case of $R \to \infty$ is considered below, but in contrast to papers [2 and 5 – 7] only one of the quantities (*) $\varepsilon = (\alpha R)^{-1/3}$, $\delta = \alpha\varepsilon$

which express the eigenvalues is assumed to be small.

The characteristic relationship of problem (1.1) can be presented in the form [2]

*) The case of small ε and arbitrary δ was inaccurately analyzed in [6], see [2 and 5].

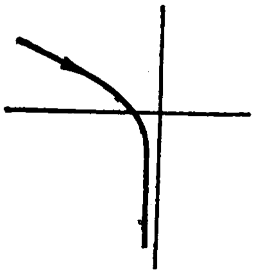

Fig. 1

$$Z_1(\delta)\, Z_2(-\delta) - Z_1(-\delta)\, Z_2(\delta) = 0 \qquad (1.2)$$

$$Z_{1,2}(\delta) = \int_{\eta_-}^{\eta_+} e^{\delta\eta} \psi_{1,2}(\eta)\, d\eta$$

$$\eta_\pm = \frac{1}{\varepsilon}\left(\pm 1 - i\frac{\alpha}{R} - c\right) \qquad (1.3)$$

Here $\psi_{1,2}$ are independent solutions of Eq.

$$i\psi'' + \eta\psi = 0 \qquad (1.4)$$

In the following use will be made of functions [2]

$$A_n(\eta) = \frac{1}{2\pi i} \int \exp(\eta z + {}^1/_3 i z^3)\, z^{n-1} dz \qquad (1.5)$$

the integration path of which is shown on Fig. 1.

We may take [2]

$$\psi_1 = A_1, \qquad \psi_2 = \omega A_1(\omega\eta) \qquad (\omega = e^{2/_3 i\pi}) \qquad (1.6)$$

as the solution of (1.4).

Substitution of (1.6), (1.5) into (1.3) and intergation over η yields

$$Z_1(\delta) = I(\delta, \eta_+) - I(\delta, \eta_-), \qquad Z_2(\delta) = I(\delta/\omega, \omega\eta_+) - I(\delta/\omega, \omega\eta_-)$$

$$I(\delta, \eta) = \frac{1}{2\pi i} \int \exp(\eta\delta + \eta z + {}^1/_3 i z^3) \frac{dz}{z+\delta} \qquad (1.7)$$

The integration path is here the same as in (1.5), and lies to the left of the pole $z = -\delta$.

As regards the eigenvalues of problem (1.1), it is known [2 and 3] that if $c = c_r + ic_i$ is an eigenvalue, then $c = -c_r + ic_i$ is also an eigenvalue, and $|c_r| \leqslant 1$

Purely imaginary eigenvalues correspond (according to [4]) to damped perturbations, provided the relation $c_i + \alpha/R < 0$ is fulfilled. It is shown in the Appendix that this relation is fulfilled if

$$\alpha R\, |c_r| \leqslant 6 \qquad (1.8)$$

It follows from this that the flow is stable for $\alpha R \leqslant 6$. In connection with this we shall consider the case of $\delta \neq 0$ on the assumption that

$$\varepsilon \to 0, \quad |c_r| \sim 1 \qquad (1.9)$$

It is shown in the Appendix that the flow is stable when

$$\delta \geqslant ({}^{27}/_{256})^{1/4} \approx 0.7 \qquad (1.10)$$

However for the sake of completeness of the picture, the case of an arbitrary δ is considered.

2. In the case of (1.9) when $c_r \sim -1$

$$|\eta_+| \sim 1/\varepsilon \to \infty, \qquad \Theta = \arg \eta_+ \approx 0$$

and the integrals (1.7) may be estimated for $\eta = \eta_+$ by the saddle-point method. The saddle-point contribution

$$z_0 = \eta^{1/2} e^{-3/4\, i\pi}$$

the integral of (1.7) is

$$N(\delta, \eta) = -\frac{\exp(\delta\eta + {}^2/_3\, \eta\, z_0 - {}^1/_4\, i\pi)}{2\sqrt{\pi}\, z_0^{1/2} (z_0 + \delta)} \left[1 + O(\eta^{-3/2})\right]$$

Here and below the argument of the power of a number is equal to the argument of the number multiplied by the exponent.

For $|\eta_+| \to \infty$ and any values of δ we have

$$I(\delta, \eta_+) = N(\delta, \eta_+) \qquad (-{}^1/_2\pi < \Theta < {}^1/_2\pi) \tag{2.1}$$

$$I(-\delta, \eta_+) = N(-\delta, \eta_+) \qquad (-{}^7/_6\pi < \Theta < {}^5/_6\pi) \tag{2.2}$$

$$I(\delta/\omega, \omega\eta_+) = N(\delta/\omega, \omega\eta_+) \qquad (-{}^{11}/_6\pi < \Theta < {}^1/_6\pi) \tag{2.3}$$

$$I(-\delta/\omega, \omega\eta_+) = N(-\delta/\omega, \omega\eta_+) + \exp(-{}^1/_3 i\delta^3) \qquad (-{}^1/_2\pi < \Theta < {}^1/_6\pi) \tag{2.4}$$

In order to obtain an estimate of (2.4) homogeneous in δ it is necessary for the integration path to lie to the right of pole $z = \delta/\omega$, hence the appropriate residue has been taken into account.

All estimates (2.1) – (2.4) hold in the domain

$$\Delta - {}^1/_2\pi \leqslant \Theta \leqslant {}^1/_6\pi - \Delta \quad (\Delta > 0)$$

From (1.7), (2.1), (2.3) we obtain

$$Z_1(-\delta) = -I(-\delta, \eta_-), \qquad Z_2(\delta) = I(\delta/\omega, \omega\eta_+)$$

which is correct to within exponentially small terms. The remaining expressions for Z depend on magnitude $\Lambda = \operatorname{Re}(-\delta\eta_+ - {}^2/_3\eta_+ z_0)$

Let $\Lambda \gg 1$. Then N is exponentially small in (2.2), and exponentially large in (2.4), so that $Z_1(\delta) = -I(\delta, \eta_-)$, $Z_2(-\delta) = I(-\delta/\omega, \omega\eta_+)$

and relation (1.2) assumes the form

$$e^{2\alpha} J(\delta, \eta_-) - He^{-2\alpha} J(-\delta, \eta_-) = 0 \tag{2.5}$$

$$J(\delta, \eta) = e^{\delta\eta} I(-\delta, \eta), \qquad H = \frac{J(\delta/\omega, \omega\eta_+)}{J(-\delta/\omega, \omega\eta_+)} = 1 + O\left(\frac{\delta}{\eta_+^{1/2}}\right)$$

If $\alpha = \text{const}$, then $\delta/\eta_+^{1/2} \sim \alpha\varepsilon^{1/2}$, and it becomes necessary to reject the terms $O(\varepsilon^{1/2})$ in (2.5).

As the result (2.5) becomes the relation (*)

$$A_0(\eta) + (\alpha \operatorname{cth} 2\alpha)\varepsilon A_{-1}(\eta)\,[1 + O(\varepsilon^{-1/2})] = 0 \tag{2.6}$$

At its limit for $\alpha \to 0$, $\varepsilon = \text{const} \ll 1$ this relation is reduced to the equality derived earlier by other means [2].

It is worth noting that this limit equality was obtained in [2] for $\alpha \neq 0$ the inaccuracy in [2] is associated with the fact that there the ratio of rejected terms $O(\delta^{2n})$ to those retained was $\sim \delta^{2n} A_{4n}(\eta_+)/A_0(\eta_+) \sim \alpha^{2n}$, which is small for small α only.

In the case of $\alpha \to \infty$, $\delta = \text{const}$ we obtain from (2.5) correct to within terms of the order of $\exp(-4\alpha)$

$$J(\delta, \eta) \equiv \frac{1}{2\pi i}\int \frac{\exp(\eta z + {}^1/_3 iz^3)}{z - \delta}\, dz \equiv \sum_{n=0}^{\infty} \delta^n A_{-n}(\eta) = 0 \tag{2.7}$$

Here the second expression for J is derived from the first by taking an integration path lying outside the circle $|z| = \delta$ (which is always possible), expanding $1/(z - \delta)$

*) Here and in the following the subscript of η_- is omitted.

in a series in δ / z, and integrating by parts.

In the deriving (2.5) and (2.6) we assumed that the above quantity $\Lambda \gg 1$.

If $\Lambda \lesssim 1$ (which is possible when $\delta^{-1} = O(\varepsilon^{1/3})$), then (2.7) is obtained from (1.2), (1.7) and (2.1) – (2.4) to within the terms $O(\exp {}^2/_3\eta_+ z_0)$. Thus, relations (2.5) and (2.7) are valid for all values of δ.

3. For $\varepsilon \to 0$ and finite α we find from (2.6) that $A_0(\eta) = 0$. The roots of this equation were analyzed in [2 and 5], and correspond to damped perturbations. It remains to consider the case of finite δ.

The roots of Eq. (2.7) of large absolute value can be investigated with the aid of the assymptotic expression

$$J(\delta, \eta) = V_+ + V_- + \exp(\eta\delta + {}^1/_3 i\delta^3) \qquad (3.1)$$

$$V_\pm = -\frac{\exp[\pm({}^2/_3\eta z_0 - {}^1/_4 i\pi)]}{2(\pi z_0)^{1/2}(z_0 \mp \delta)}, \qquad z_0 = \eta^{1/2} e^{-3/4 i\pi}$$

$$(|\eta| \to \infty, \quad -{}^3/_2\pi \leqslant \arg\eta \leqslant -{}^1/_2\pi - \Delta, \quad \Delta > 0)$$

When deriving estimate (3.1) homogeneous in δ, the integration path is taken to the right of pole $z = \delta$; The quantities $V_\pm$ are the contributions of the saddle-points $z = \pm z_0$, and the third term of (3.1) is the residue of point $z = \delta$.

It is convenient to present (3.1) in the form (3.2)

$$J(\delta, \eta e^{-2/3 i\pi}) = \sqrt{\pi}\,\eta^{-1/4}(\eta + \delta^2\beta^2)\exp({}^1/_3 i\delta^3 - \delta\eta\beta) - \cos x + \beta\delta\eta^{-1/4}\sin x$$

$$x = {}^1/_4\pi + {}^2/_3\eta^{3/2}, \qquad \beta = e^{-1/3 i\pi} \qquad (-{}^1/_3\pi \leqslant \arg\eta \leqslant {}^2/_3\pi - \Delta)$$

The flow is unstable if $\arg\eta > 1/6\pi$ for any of the roots of Eq. $J = 0$. Assuming

$$\eta^{1/2} = re^{i\varphi}, \qquad \rho = \delta / r$$

and assuming that φ, ρ are small, we obtain from (3.2)

$$\cos x = \sqrt{\pi}\, r^{3/2} e^{-\delta\beta r^2} \equiv F(r, \delta), \quad x = ({}^1/_4\pi + {}^2/_3 r^3) + i(2r^3\varphi) = a + ib \quad (3.3)$$

For large F it can be assumed that $\cos x = 1/2\exp(\mp ix)$. Here and below the upper sign corresponds to the "upper" roots in which $\varphi > 0$.

A comparison of the amplitudes and phases of magnitudes appearing in (3.3) yields

$$b = \pm\ln 2|F|, \qquad a \pm \psi = 2\pi n \qquad (\psi = {}^1/_2\delta r^2,\ \delta \geqslant 0,\ n \geqslant 1) \qquad (3.4)$$

From (3.4) follows

$$r \approx r_0 \mp {}^1/_4\delta, \qquad r_0^3 = 3\pi(n - {}^1/_8), \qquad \varphi \approx \pm {}^1/_2\, r_0^{-3}\ \text{In}\ 2|F(r_0, \delta)| \quad (3.5)$$

Expressions (3.5) show that with increasing δ the two angles decrease, while the radius increases at the lower root and decreases at the upper root.

Relationships (3.4), (3.5) are not valid when

$$\delta \sim \delta_1 = 3^{-1/3} r_0^{-2}\ \text{In}\ (\pi r_0^3)$$

where $F \sim 1$. The value of δ_1 is defined by the equality $|F| = 1$.

For $\delta \sim \delta_1$ the quantity $\psi \approx {}^1/_2\delta_1 r_0^2 = \psi_1$ is large, hence the number of upper roots in the disk $r < r_0$ exceeds that of the lower roots by $2\psi_1$ the total number of roots is approximately the same as in the case $\delta = 0$.

For $\delta = \infty$ Eq. (3.3) has the solution $\varphi = 0$

$$x_m = a(r_m) == {}^1/_2\pi + \pi m \qquad (m \gg 1) \tag{3.6}$$

For the small divergence $\chi = x - x_m$ we obtain from (3.3)

$$\chi = (-1)^{m+1} F(r_m, \delta) \qquad (\delta \to \infty) \tag{3.7}$$

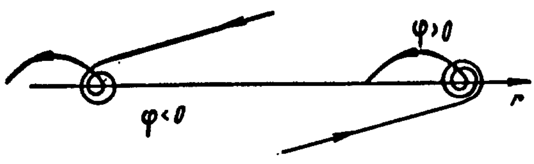

Fig. 2

It follows from this that points (3.6) are stable focal points; spirals $\chi(\dot{\delta})$ wind counterclockwise around the focal points.

For $\delta > \delta_1$ each of the roots $\eta(\delta_1)$ winds around one of the nearest focal points at which $r_m \approx r(\delta_1)$.

From this we obtain (*) $\quad m \approx 2n - {}^1/_2 \mp \psi_1/\pi$

The winding stops when

$$\delta \sim \delta_2 \approx 3^{-1/2} r_0^{-2} \ln ({}^3/_4 \pi r_0^9)$$

i.e. when the last term of (3.2) becomes comparable to F. The value of δ_2 is defined by equality $\rho = |F|$. The number of loops is approximately equal to (**)

$$(\psi_2 - \psi_1)/(2\pi) \approx r_m^2(\delta_2 - \delta_1)/(4\pi) \approx (\ln r_m{}^3)/(2\pi\sqrt{3})$$

In the domain E $(\delta_2 < \delta < \infty)$ the residue in (3.1), (3.2) may be neglected (***), and (3.3) can be written in the form

$$e^{-2ix} = (i\beta\rho + 1)/(i\beta\rho - 1)$$

Multiplication and division of this equation by its complex conjugate yields

$$e^{4b} = \frac{1+\rho+\rho^2}{1-\rho+\rho^2}, \qquad e^{4ia} = \frac{1-\rho^2-i\sqrt{3}\rho}{1-\rho^2+i\sqrt{3}\rho} \tag{3.8}$$

It follows from (3.8) that the quantity a decreases in the domain E from a value $\approx x_m$ to the value πm, while angle

$$\varphi = {}^1/_8 r^{-3} \ln[(1+\rho+\rho^2)/(1-\rho+\rho^2)] \tag{3.9}$$

increases from zero to its maximum value for $\rho \approx 1$, and then decreases to zero.

On the basis of the foregoing, we can expect that for large n the pair of roots will vary with increasing δ in the manner shown on Fig. 2.

It will be seen from (3.5), (3.9) that for large n the angles $\varphi \ll 1$, hence the respective perturbations are damped.

The first pair of roots $\eta = \mu + i\nu$ of Eq. (2.7) was analyzed numerically. The method of computation is given in the Appendix, and the results are shown on Fig. 3. It follows from Fig. 3 and inequality (1.10) that the first pair of roots defines damped perturbations, so that the flow is stable for $\varepsilon \to 0$ and any α.

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*) It can be expected that for small n (when $\psi_1 \sim 1$) we have $m = 2n - {}^1/_2 \mp {}^1/_2$).

**) It can be expected that for small m (when $\psi_2 \sim 1$) the number of loops will be zero, i.e. the roots will tend aperiodically to points (3.6).

***) Asymptotic expressions in [6] do not take into account the residue, hence the results obtained in that paper hold within the domain E only.

4. The characteristic relationship for the case of $\delta = 0$, $\varepsilon > 0$ is derived from (1.2), (1.7) by taking the limit as $\alpha \to 0$, $\varepsilon = \mathrm{const}$. This relation can also be written in the form (cf. [2])

$$J \equiv A_0(\eta) - A_0(T) - \omega\,[A_2(\eta) - A_2(T)]\,\frac{A_0(\omega T) - A_0(\omega\eta)}{A_2(\omega T) - A_2(\omega\eta)} = 0 \qquad (4.1)$$

$$(T = 2/\varepsilon + \eta, \quad \omega = e^{2/3\, i\pi})$$

Equality (2.6) in which $\alpha = 0$ can be derived from this as $\varepsilon \to 0$. For arbitrary ε the roots η were obtained by numerical methods (see Appendix). The results of computation of the first pair of roots are shown on Fig. 4. The values of ν, as calculated in [8] for the first pair of roots, are shown by a dotted line. Damping of the respective perturbations is apparently slowest with any ε.

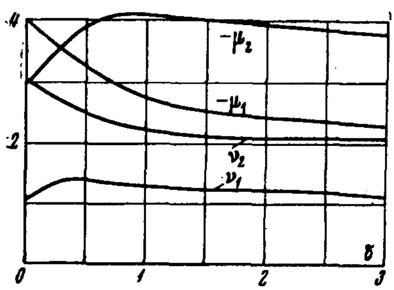

Fig. 3

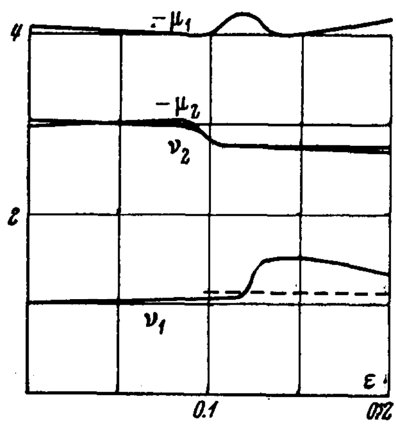

Fig. 4

If this is so, then the results of [8] and Fig. 4 imply that the flow is stable for $\alpha = 0$ and any ε. This and the results adduced in Section 4 show that instability is possible for finite values of R only.

5. Appendix. The characteristic relation obtained in [4] for problem (1.1), after substitution in its equation of $\varkappa y$ for y which figures there explicitly, was used as the initial relation in deriving (1.8). This relationship is of the form [4]

$$\Delta \equiv \sum_{p=1}^{\infty} \Big(\sum_{m=0}^{p-1} \sum_{n=1}^{p-m} a_{p,m,n} d^m \gamma^{2n-1} \Big) k^{2p+2} \qquad (A.1)$$

where only such n are taken for which the number $\nu = 1/3\,(p - n - m)$ is an integer, and

$$d = -ic + \alpha/R, \qquad \gamma \doteq \alpha/k, \qquad a = \alpha R/k^3$$

$$a_{p,m,n} = \frac{2^{2n+1}}{(2p+2)!}(4a)^{p-n-\nu} \sum_{q=0}^{\nu} (-1)^q \binom{2p-2q}{2n-1} \frac{(p-n-q)!}{3^{\nu-q}(\nu-q)!\,m!}$$

Assuming in (A.1) that $k=1$, and changing the summation sequence, we obtain

$$\Delta \equiv \sum_{m=0}^{\infty} D^m A_m = 0, \qquad D = \alpha R d, \; A_m = 8\alpha \sum_{n=\nu=0}^{\infty} B_{m,n,\nu} (4\alpha^2)^n (16/3\,\alpha^2 R^2)^\nu \qquad (A.2)$$

$$B_{m,n,\nu} = \frac{4^m}{m!\,(2n+1)!\,(6\nu+2n+2m+4)!} \times$$

$$\times \sum_{q=0}^{\nu} (-3)^q \frac{(3\nu+m-q)!\,(6\nu+2n+2m+2-2q)!}{(\nu-q)!\,(6\nu+2m+1-2q)!}$$

Coefficients A satisfy inequality

$$A_{m-1} > m^2 A_m \qquad (m \geqslant 1) \tag{A.3}$$

because it is satisfied by the quantities $B_{m,n,\nu}$ for any n, ν. This can be verified by noting that $B_{m,n,\nu}$ is a sum with an alternating sign and terms monotonically decreasing in absolute value. Such a sum is not greater than its first term $b_{m,n,\nu}$, and not smaller than the sum of its first two terms. Taking this into account, we obtain

$$B_{m-1,n,\nu} - m^2 B_{m,n,\nu} \geqslant b_{m-1,n,\nu}\left[1 - \frac{3\nu(6\nu+2m-1)}{(3\nu+m+n)(6\nu+2m+2n-1)}\right] -$$

$$- m^2 b_{m,n,\nu} \geqslant b_{m-1,n,\nu}\left[1 - \frac{3\nu}{3\nu+m} - \frac{m}{3\nu+m+1/2}\right] > 0$$

Let us assume that $D = \gamma + i\Omega = |D|\,e^{i\varphi}$ for certain values of the parameters becomes purely imaginary, i.e. $\gamma = 0$. We then obtain from (A.2)

$$\Delta_r \equiv \sum_{m=0}^{\infty} (-1)^m \Omega^{2m} A_{2m} = 0, \qquad \Delta_i \equiv \sum_{m=0}^{\infty} (-1)^m \Omega^{2m+1} A_{2m+1} = 0 \tag{A.4}$$

From this and (A.3) follows that for $|\Omega| = \alpha R\,|c_r| \leqslant 6$ the terms of sum Δ_i decrease monotonically in absolute value, so that $\Delta_i > 0$. This means that if $|\Omega| \leqslant 6$ then $\gamma = \alpha R\,(c_i + \alpha|R) \neq 0$ It is readily seen that $\gamma < 0$, as this is true for $R = 0$ (when roots D are real [3]), and functions $D\,(R)$ are continuous.

Other root estimates are derived with the aid of inequality [9]

$$\sum_{m=0}^{\infty} a_m e^{im\varphi} \neq 0$$

which is fulfilled for any φ, if coefficients a_m are positive and decrease monotonically with increasing m. From this and (A.3) follows

$$\frac{d^n \Delta}{dD^n} = \sum_{m=0}^{\infty} \frac{(m+n)!}{n!} A_{m+n}\,|D|^m e^{im\varphi} \neq 0$$

if $|D| < n+1$. This means that there are no roots in the domain $|D| < 1$, while in the domain $\gamma \geqslant -n-1$ the number of real roots does not exceed n In a similar manner we obtain from (A.4)

$$\frac{d^n(\Delta_i/\Omega)}{d(-\Omega^2)^n} = \sum_{m=0}^{\infty} (-\Omega^2)^m \frac{(m+n)!}{m!} A_{2m+2n+1} > 0$$

if $|\Omega| \leqslant 2\,(2n+3)\sqrt{n+1} = \Omega_n$. This shows that in the domain $0 < |\Omega| < \Omega_n$ there are not more than n purely imaginary roots $D = i\Omega$.

The following inequality [1] was used in the deviation of (1.10)

$$\sigma = \alpha R c_i\,(I_1^2 + \alpha^2 I_0^2) \leqslant \alpha R I_0 I_1 - (I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)$$

where

$$I_n^2 = \int_{-1}^{1} dy \left| \frac{d^n\varphi}{dy^n} \right|^2$$

and φ is the solution of problem (1.1). Because for any real constants $\zeta, \varkappa$

$$L = \int_{-1}^{1} |\chi\varphi + \zeta\varphi' + \varphi''|^2 dy = \chi^2 I_0^2 + (\zeta^2 - 2\chi) I_1^2 + I_2^2 > 0$$

then (*)

$$\sigma < \alpha R I_0 I_1 - I_1^2 (2\alpha^2 + 2\varkappa - \zeta^2) - I_0^2 (\alpha^4 - \varkappa^2) \leqslant 0$$

if

$$(\alpha R)^2 \leqslant 4 (2\alpha^2 + 2\chi - \zeta^2) \quad (\alpha^4 - \chi^2)$$

and every multiplier in the right-hand side is positive.

The right-hand side of the last inequality attains its maximum for $\zeta = 0$, $\chi = {}^1/_3\,\alpha^2$ and coincides with (1.10).

The iteration process [10]

$$\eta_{k+1} = \eta_k - F(\eta_k, \delta) \qquad (k \geqslant 0)$$

was used for machine computation of the roots of Eq. (2.7) where

$$F(\eta, \delta) = \frac{J}{J'} + \frac{J''}{2J'}\left(\frac{J}{J'}\right)^2 + \left[{}^1/_2\left(\frac{J''}{J'}\right)^2 - \frac{J'''}{6J'}\right]\left(\frac{J}{J'}\right)^3 \tag{A.5}$$

$$J^{(n)} = \partial^n J(\eta, \delta) / \partial\eta^n$$

and η_0 is sufficiently close to $\eta(\delta)$. The already known value of $\eta(\delta_0)$ with δ_0 close to δ was taken as η_0 Values of η for $\delta = 0$ were taken from [2].

The series expansion of (2.7) and the relationship [2]

$$iA_{n+3} + \eta A_{n+1} + nA_n = 0 \tag{A.6}$$

were used for computing J.

In order to utilize (A.6) it is sufficient to find coefficients $A_n(\eta)$ for $n = 0,1,2$. These were determined with the aid of equalities

$$A_n(\eta) = \sum_{m=0}^{\infty} \frac{\eta^m}{m!} A_{n+m}(0) \tag{A.7}$$

We note that the method of saddle-point makes it possible to derive

$$|A_{1+n}| = f_1 |n|^{-1/6} \exp {}^1/_3\, n \; [\ln|n| - 1 + |\eta n^{-2/3}| f_2] \qquad (f_{1,2} \sim 1, |n| \to \infty)$$

hence expansions (A.7), (2.7) are everywhere convergent.

Equality (A.7) is obtained from (1.5) by the series expansion of $\exp \eta z$, and term by term integration. Coefficients $A_n(0)$ are determined with the aid of (A.6), provided the first three coefficients for $n = 0,1,2$ are known. The latter are

$$A_0(0) = {}^1/_3, \quad A_n(0) = 3^{3n-1} \frac{\Gamma({}^1/_3\, n)}{2\pi i} [e^{-1/3 i\pi n} - e^{-2/3 i\pi n}] \qquad (n > 0)$$

It is convenient to compute series $A_{0,1,2}(\eta)$ simultaneously. The derivatives of J are defined by equality

$$J^{(n)} = A_n + J^{(n-1)}\delta \qquad (n \geqslant 1)$$

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*) An incorrect expression was used in [1] for the integral of the L type, and consequently the relevant stability conditions derived there are incorrect.

The roots of Eq. (4.1) were calculated by the Newton method (Formula (A.5) in which $F = J / J'$). Series (A.7) were used for the determination of F. The exact Expression (4.1) was used with $\beta = |T|^{2/3} (1/\sqrt{3} - T_i / T_r) < 10$. Negative magnitudes A were rejected in (4.1) for $10 \leqslant \beta < 40$. For $\beta \geqslant 40$ Expression (2.6) in which $\alpha = 0$ was used for J. Computations were commenced with $\varepsilon = 0$, and the initial values were taken from [2].

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